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The Hill determinant approach to the Coulomb plus linear confinement

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Abstract. An application of the so-called Hill determinant method to the bound-state eigenvalue problem in the elementary quarkonium potential $V(r)=-a/r+br$ is described, proved and illustrated for a few examples. An improvement of the method, which is based on an extended continued-fraction formulation of the eigenvalue condition, is also proposed.

1. Introduction and summary

For a class of potentials, the Schrödinger eigenvalue problem may be solved, in a formal analogy between the finite- and infinite-dimensional linear matrix equations, by the so-called Hill determinant technique (Biswas *et al* 1971). Its rigorous mathematical foundation is usually missing (cf, e.g., Ginsburg (1982) and the references given therein) and we may often encounter contradictions (for details, cf, e.g., Chaudhuri and Mukherjee 1984, Flessas 1982, Znojil 1982, 1983a).

In the recent reformulation of the above method (Znojil 1983b) applicable to the potentials $v(r)=\Sigma_{n/m>-2}\,\gamma_n r^{n/m}$ we are permitted to reinterpret the usual asymptotic boundary conditions or normalisation requirement

$$\|\psi\|<\infty \tag{1.1}$$

as an appearance of the Hill determinant zero. Unfortunately, an unpleasant inequality must also be imposed on the couplings γ_n.

For the particular example

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\frac{a}{r}+br\right)\psi(r)=E\psi(r) \qquad l=0,1,\ldots,b>0 \tag{1.2}$$

with the Coulomb plus linear potential, the latter restriction is not acceptable on physical grounds (cf, e.g., the important applications of (1.2) in the physics of quarkonia: Quigg and Rosner (1979), etc).

In the present resolution of the above puzzle, we notice that, in contrast to the assumptions of Znojil (1983b), the potential does not contain the $r^{\pm 1/2}$ components. Hence, we may use a simplified power series ansatz

$$\psi(r)=r^{l+1}\,e^{-\beta r}\sum_{m=0}^{\infty}p_m r^m \qquad \beta>0. \tag{1.3}$$

We shall see below that such a form of ansatz enables one to avoid the contradictions.

Our main result will be a rigorous proof of the applicability of the Hill determinant technique to equation (1.2), i.e. a proof of the equivalence of equation (1.1) to the Hill determinant prescription

$$P_n(E)=0 \qquad n\gg 1 \tag{1.4}$$

in the limit $n\to\infty$.

2. Power series ansatz and its convergence

When we insert (1.3) into (1.2), we obtain a set of relations

$$\begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ d_2 & c_2 & a_2 & b_2 & \\ & d_3 & c_3 & a_3 & b_3 \\ & & & \cdots & \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \vdots \end{pmatrix}=0 \tag{2.1}$$

$$a_n=-2\beta(n+l+1)+a \qquad c_{n+1}=\beta^2+E$$

$$b_n=(n+1)(n+2l+2) \qquad d_{n+2}=-b,\ n=0,1,\ldots$$

i.e. recurrences

$$p_{n+1}=-(1/b_n)(d_np_{n-2}+c_np_{n-1}+a_np_n) \qquad p_{-1}=p_{-2}=0 \tag{2.2}$$

with an obvious solution

$$p_{n+1}=\frac{(-1)^{n+1}p_0}{b_0b_1\ldots b_n}\det Q^{(n)} \qquad Q^{(n)}=\begin{pmatrix} a_0 & b_0 & & \\ & \cdots & & \\ & & \cdots & \\ & d_n & c_n & a_n \end{pmatrix}. \tag{2.3}$$

Hence, with an arbitrary normalisation p_0, we may interpret (1.3) as a regular solution $\psi(r)$ with the determinantal definition (2.3) of coefficients.

We may postulate a formal decomposition

$$Q^{(\infty)}=\begin{pmatrix} 1 & & & \\ u_1 & 1 & & \\ v_2 & u_2 & 1 & \\ & v_3 & u_3 & 1 \\ & & \ddots & \end{pmatrix}\begin{pmatrix} h_0 & b_0 & & \\ & h_1 & b_1 & \\ & & h_2 & b_2 \\ & & & \ddots \end{pmatrix} \tag{2.4}$$

where $a_0=h_0$, $c_1=u_1h_0$, $a_1=u_1b_0+h_1$ and

$$a_{n+2}=u_{n+2}b_{n+1}+h_{n+2}$$

$$c_{n+2}=v_{n+2}b_n+u_{n+2}h_{n+1}$$

$$d_{n+2}=v_{n+2}h_n \qquad n=0,1,\ldots.$$

Then we may eliminate $u_{n+2}=(c_{n+2}-v_{n+2}b_n)/h_{n+1}$ and $v_{n+2}=d_{n+2}/h_n$. The remaining recurrences

$$h_{n+2}=a_{n+2}-\frac{c_{n+2}b_{n+1}}{h_{n+1}}+\frac{d_{n+2}b_nb_{n+1}}{h_nh_{n+1}} \tag{2.5}$$

together with the initial values

$$h_0 = a_0 \qquad h_1 = a_1 - c_1 b_0/a_0 \tag{2.6}$$

define the factorisation (2.4) unambiguously. They also enable us to rewrite (2.3) as a product

$$p_{n+1} = p_0 \frac{(-1)^{n+1}}{b_0 b_1 \dots b_n} h_0 h_1 \dots h_n \tag{2.7}$$

suitable for analysis of p_{n+1} at large $n \gg 1$.

The asymptotic form of (2.5) implies oscillations, $h_{n+2} \approx d_n b_n b_{n+1}/h_n h_{n+1}$. Their asymptotic damping

$$h_n^{(0)} \approx h_n \approx h_{n+1} \approx h_{n+2} \qquad n \gg 1 \tag{2.8}$$

is analysed in appendix 1. In general, the functions

$$h_n^{(0)} \approx -2\beta n - (\beta^2 + E)\frac{n^2}{h_n^{(0)}} - b\frac{n^4}{h_n^{(0)2}} \approx -\tilde{\rho} n^{4/3} \qquad \tilde{\rho}^3 = b \tag{2.9}$$

may become complex (Im $\tilde{\rho} \neq 0$), but the estimate

$$\frac{p_{n+1}}{p_n} \approx -\frac{h_n^{(0)}}{b_n} = O(n^{-2/3}) \qquad n \gg 1 \tag{2.10}$$

may be used for large n and all energies. It implies that the power series (1.3) will converge and satisfies the differential equation (1.2) for all $r \geq 0$.

In the origin, the solution (1.3) is regular. Its asymptotic behaviour ($r \gg 1$) remains ambiguous

$$\psi(r) \approx c_1 \exp(\tfrac{2}{3}\sqrt{b}\, r^{3/2}) + c_2 \exp(-\tfrac{2}{3}\sqrt{b}\, r^{3/2}) \tag{2.11}$$

and becomes compatible with the requirement of physical normalisability (1.1) if and only if

$$c_1 = c_1(E) = 0 \tag{2.12}$$

i.e. exactly at the particular bound-state energies $E = E_m$, $m = 0, 1, \dots$.

3. Normalisation requirement and a rigorous proof of the Hill determinant method

Lemma 1. The asymptotic solution p_n of recurrences (2.1) corresponds to the exponentially growing wavefunction $\psi(r)$ for almost all energies.

Proof. An iteration of (2.10) gives

$$p_{n+k} = \frac{(-\tilde{\rho})^k p_n}{[\Gamma(n+k)/\Gamma(n)]^{2/3}} + \text{small corrections} \qquad n \gg k^3 \gg 1. \tag{3.1}$$

We may simplify this relation by means of the Stirling formula:

$$[\Gamma(n+k)/\Gamma(n)]^{2/3} \approx \Gamma(n + \tfrac{2}{3}k)/\Gamma(n) \qquad n \gg 1. \tag{3.2}$$

Due to the results of appendix 1, we have $(-\tilde{\rho})>0$ and may replace the summation approximately by an integral in (1.3):

$$\sum_{k=k_0}^{\infty} p_k r^k \approx \int_{k_0}^{\infty} p_k r^k \, \mathrm{d}k \qquad k_0 \gg 1. \tag{3.3}$$

Finally, we may change the variables $2k/3=m$ and obtain the leading-order formula

$$\psi(r) \approx \ldots + \mathrm{e}^{-\beta r} p_n r^{n+l+1} \sum_m \frac{(\tilde{\rho} r)^{3m/2} \Gamma(n)}{\Gamma(n+m)} \approx p_n \exp|\sqrt{b} r^{3/2}| \qquad n, r \gg 1 \tag{3.4}$$

after some straightforward error-estimate mathematics.

We see that the growing behaviour estimate (3.4) of $\psi(r)$ agrees well with our expectations (2.11). An important property of (3.1) and (3.4) is a factorisation of $\psi(r)$, $r \gg 1$ into a positive function (the exponential also contains implicitly the errors) multiplied by the coefficient $p_n = p_n(E)$, $n = n_0$. It is to be represented by the determinant (2.3) and we arrive rigorously at the desired result.

Theorem. The spectra of energies defined as zeros of the Hill determinants

$$\det Q^{(N)} = 0 \qquad N \gg 1 \tag{3.5}$$

coincide with the complete physical binding energy spectrum in the limit $N \to \infty$.

Proof. The relation (3.4) is to be considered as a function of energy E. As long as $p_N = p_N(E)$ is a polynomial, the approximate asymptotics of $\psi(r)$, $r \gg 1$ will change sign precisely at the zeros of $p_N(E)$. Thus, in accord with the standard oscillation theorems valid for the general Sturm–Liouville problem of the type (1.2) (Ince 1956), a new node of $\psi(r)$ at large r may be interpreted as an appearance of a new bound state. Hence, the corresponding energy may be approximated by the zero of (3.5) at a sufficiently large $N \gg 1$.

4. Simplification of Hill determinants via the extended continued fractions

In the practical applications, the zeros of determinants (3.5) may be found numerically via the factorisation

$$Q^{(N)} = \begin{pmatrix} 1/f_0 & b_0 & & \\ & 1/f_1 & b_1 & \\ & & \ddots & \\ & & & 1/f_N \end{pmatrix} \begin{pmatrix} 1 & & & & \\ \omega_1 & 1 & & & \\ z_2 & \omega_2 & 1 & & \\ & z_N & \cdot & \omega_N & 1 \end{pmatrix} \tag{4.1}$$

$$a_n = 1/f_n + b_n \omega_{n+1} \qquad c_{n+1} = \omega_{n+1}/f_{n+1} + b_{n+1} z_{n+2}$$

$$d_{n+2} = z_{n+2}/f_{n+2} \qquad n = 0, 1, \ldots$$

(Znojil 1983b). Indeed, the definitions

$$\begin{aligned} z_{n+2} &= f_{n+2} d_{n+2} \\ w_{n+1} &= f_{n+1} c_{n+1} - f_{n+1} b_{n+1} f_{n+2} d_{n+2} \end{aligned} \tag{4.2}$$

and basic recurrence

$$1/f_n^{(N)} = a_n - b_n f_{n+1}^{(N)} c_{n+1} + b_n f_{n+1}^{(N)} b_{n+1} f_{n+2}^{(N)} d_{n+2}$$

$$n = 0, 1, \ldots, N \qquad f_{N+1}^{(N)} = f_{N+2}^{(N)} = 0 \tag{4.3}$$

convert (4.1) into an algebraic identity.

The main merit of the factorisation lies in its consequence

$$\det Q^{(N)} = \prod_{k=0}^{N} 1(f_k^{(N)})^{-1} \qquad N \geq 0. \tag{4.4}$$

When we assume

$$1/f_k^{(N)} \neq 0 \qquad N \geq k > M \geq 0 \tag{4.5}$$

we may convert the Hill determinant secular equation (3.5) into the abbreviated extended-continued-fractional (ECF, Znojil 1981) prescription

$$\prod_{k=0}^{M} 1/f_k^{(N)} = 0 \qquad N \gg M. \tag{4.6}$$

Let us show now that we may use always $M = 1$ in (4.6) and ignore the assumption (4.5) completely.

Assuming the opposite, we shall have such $M > 1$ that

$$1/f_{M+1}^{(N)} \to 0 \qquad \text{i.e. } f_{M+1}^{(N)} \to \infty.$$

Then equation (4.3) with $n = M$ implies that

$$1/(f_M^{(N)} f_{M+1}^{(N)}) \to b_M(-c_{M+1} + b_{M+1} f_{M+2}^{(N)} d_{M+2}). \tag{4.7}$$

In general, it is a definite ($0 \times \infty$ type) non-zero factor computable by recurrences (4.3) for any $M < N$. Obviously, the pair of factors (4.7) may be cancelled out of (4.6) and the assumption (4.5) proves redundant. We could easily put $M = 0$ in (4.6).

Of course, it may also happen that the right-hand side expression in (4.7) happens to be zero. Then, we get $1/f_M^{(N)} = a_M < \infty$ and, recalling (4.3) once more (with $n = M - 1$, i.e. for all $M \leq N$ but $M = 0$), we get the conclusion that

$$1/(f_{M-1} f_M f_{M+1}) \to b_{M-1} b_M d_{M+1} \qquad M \geq 1. \tag{4.8}$$

Thus, we come to the final result—equation (4.6) may be given the simple ECF form $1/(f_0^{(N)} f_1^{(N)}) = 0$, $N \to \infty$, i.e.

$$a_0/f_1^{(\infty)} = b_0(c_1 - b_1 f_2^{(\infty)} d_2) \tag{4.9}$$

in the $N \to \infty$ limit.

Numerically, the ECF convergence of the transition $N \to \infty$ proves to be rather slow—this follows from the existence of oscillations which are only slowly damped. A more thorough account of this phenomenon being given in appendix 1, we may only conclude that a 'smooth-ECF' domain with

$$f \approx f_m^{(\infty)} \approx f_m^{(M)} \approx f_{m+1}^{(M)} \approx f_{m+2}^{(M)} \qquad M \gg m \gg 1 \tag{4.10}$$

should be used as a means of accelerating the convergence. Indeed, an algebraic specification of the ECF approximation (4.10) is quick and efficient. The 'fixed point expansion' (Znojil 1983b) result is to be used in place of the trivial ECF initialisation in (4.3). In the leading-order approximation, recurrences (4.3) imply that the fixed point value (4.10) satisfies the cubic algebraic equation

$$b_n b_{n+1} d_{n+2} f^3 - b_n c_{n+1} f^2 + a_n f - 1 = 0 \qquad n \gg 1. \tag{4.11}$$

Hence, it is sufficient to put $a_n \sim -2\beta n$, $b_n \sim n^2$, $c_n = \beta^2 + E$, $d_n = -b$ and to analyse the form of corrections by means of the Cardono formula.

5. Quarkonium spectra

As an illustration of the applicability of the present method, let us compute masses of the quark-antiquark $c\bar{c}$ and $b\bar{b}$ bound states. The corresponding eigenvalue problem (1.1), i.e. the Schrödinger equation

$$H(m, a, b)R(r) = ER(r) \qquad \|R(r)\| < \infty \tag{5.1}$$

with the mass-dependent Hamiltonian

$$H(m, a, b) = -\frac{1}{mr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{a}{r} + br + \frac{l(l+1)}{mr^2} \qquad b > 0$$

may be simplified by the scale transformation

$$H(m, a, b) = \frac{1}{m}H(1, ma, mb) = ma^2 H(1, 1, \mu)$$
$$\mu = b/m^2a^3. \tag{5.2}$$

Thus, we shall restrict ourselves to $H(1, 1, b)$ and postulate a solution of the form (1.3)

$$R(r) = r^l \exp(-\beta r) \sum_{n=0}^{\infty} p_n r^n \tag{5.3}$$

valid in the region $0 \leqslant r < \infty$.

Since the zeros of det $Q^{(N)}$ determine the energy eigenvalues $E_{nl}(1, a, b)$ of the eigenvalue problem, the bound-state masses of the $q\bar{q}$ system are given by the formula

$$M_{nl}(q\bar{q}) = E_{nl}(m_q, a, b) + 2m_q - V_0.$$

The free parameter μ is to be adjusted so as to give an agreement between an experimentally observed value of the ratio (3s − 1s)/(2s − 1s) and a value determined from (5.1).

Table 1 shows the dependence of

$$\frac{M_{30} - M_{10}}{M_{20} - M_{10}} = \frac{E_{30}(1, 1, \mu) - E_{10}(1, 1, \mu)}{E_{20}(1, 1, \mu) - E_{10}(1, 1, \mu)} \tag{5.4}$$

Table 1. The dependence of the ratio $[E_{30}(1, 1, \mu) - E_{10}(1, 1, \mu)]/[E_{20}(1, 1, \mu) - E_{10}(1, 1, \mu)]$ on the parameter μ.

μ	1s	2s	3s	$\frac{3s-1s}{2s-1s}$
0	−0.250	−0.062 5	−0.027 78	1.1852
0.025	−0.180 45	0.149 63	0.320 58	1.5179
0.050	−0.117 82	0.310 43	0.561 38	1.5860
0.075	−0.059 46	0.451 42	0.768 46	1.6206
0.1	−0.004 18	0.580 31	0.955 79	1.6424
0.125	0.048 71	0.700 63	1.129 50	1.6579

Table 2. Bound-state spectrum of the $c\bar{c}$ system (in GeV). Experimental and Dirac bound states are taken from Barik and Barik (1981).

State	Hill	Dirac	Experiment
1s	3.0970	3.097	3.097
2s	3.6860	3.6700	3.686
3s	4.0311	4.0303	4.030
4s	4.3075	4.2761	—
5s	4.5500	4.4680	4.417
1p	3.5917	3.5208	3.521
2p	3.9460	3.9156	—
1d	3.8411	3.8038	3.772

Table 3. Bound-state spectrum of the $b\bar{b}$ system (in GeV). Experimental and Dirac bound states are taken from Barik and Barik (1981).

State	Hill	Dirac	Experiment
1s	9.4336	9.4336	9.4336
2s	9.9944	10.0147	9.9944
3s	10.3230	10.3547	10.3231
4s	10.5861	10.5976	10.5476
5s	10.8170	10.7872	—
1p	9.9046	9.8517	—
2p	10.2420	10.2414	—
1d	10.1421	10.1310	—

on μ. We have chosen $\mu=0.05$, which is in fairly good agreement with experimentally found values of (3s−1s)/(2s−1s), namely 1.5840 for the $c\bar{c}$ system and 1.5861 for the $b\bar{b}$ system. Taking 1s and 2s as an input we obtain

(i) $c\bar{c}$

$$M_{nl}(m_c m_{\bar{c}})=1.375\,3649\;E_{nl}(1,1,0.05)+3.259\,045 \tag{5.5}$$

(ii) $b\bar{b}$

$$M_{nl}(m_b m_{\bar{b}})=1.309\,5155\;E_{nl}(1,1,0.05)+9.587\,887. \tag{5.6}$$

In deriving these equations we do not need the values of the quark masses.

Tables 2 and 3 show the bound-state spectra for the $c\bar{c}$ and $b\bar{b}$ systems, respectively, in GeV. The Dirac bound states are taken from Barik and Barik (1981) and they may serve as estimates of relativistic corrections. We infer from table 3 that the corrections are rather small. For the 5s state the correction is only 1.8% for the $c\bar{c}$ system and 0.3% for the $b\bar{b}$ system.

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Appendix 1. Outline of the proof of the real-fixed-point approximation (2.8)

The coefficients in (2.4) are real. Since $a_n \approx -2\beta n + O(1)$, $b_n \approx n^2 + O(n)$ and $c_n = \beta^2 + E$, $d_n = -b$, the real fixed point (2.8) or (2.9) has the value

$$h_n^{(0)} \approx -|(bn^4)^{1/3}| \qquad n \gg 1 \tag{A1.1}$$

which may be generated and/or improved by the straightforward algebra analogous to that described in Znojil (1981). It remains for us to analyse the convergence of h_n to (A1.1) with the special initial conditions (2.6).

After the first $n = N \gg 1$ steps, we get some quantities

$$h_n = -\tilde{\rho} n^{4/3} / \varphi_n \qquad \tilde{\rho} = |b^{1/3}|. \tag{A1.2}$$

They have to satisfy the asymptotic form of the recurrences (2.5)

$$\frac{1}{\varphi_{n+2}} = \frac{2\beta}{\tilde{\rho}} n^{-1/3} - \frac{\varepsilon}{\tilde{\rho}^2} \varphi_{n+1} n^{-2/3} + \varphi_n \varphi_{n+1} \qquad \varepsilon = \beta^2 + E. \tag{A1.3}$$

When we replace $\varphi_n - \varepsilon n^{-2/3}/\tilde{\rho}$ by $\tilde{\varphi}_m$, $m = n$, $n+1$, $n+2$ and assume that $\tilde{\varphi} \approx O(1)$, we may convert (A1.3) into the relations

$$\tilde{\varphi}_{n+2} = \frac{1}{x + \tilde{\varphi}_n \tilde{\varphi}_{n+1}}$$
$$x = \frac{2\beta}{\tilde{\rho}} n^{-1/3} + \frac{2\varepsilon}{\tilde{\rho}^2} n^{-2/3} + O\left(\frac{1}{n}\right). \tag{A1.4}$$

When we neglect x in (A1.4), we may simply put $\tilde{\varphi}_n \tilde{\varphi}_{n+1} \tilde{\varphi}_{n+2} = 1$, i.e.

$$\tilde{\varphi}_{3m} = R \qquad \tilde{\varphi}_{3m+1} = \rho R \qquad \tilde{\varphi}_{3m+2} = 1/(\rho R^2) \qquad 0 \ll m = m_0, m_0+1, \ldots \tag{A1.5}$$

and see that the sequence $\tilde{\varphi}_n$ oscillates. These oscillations are damped by the influence of x, which follows from the geometric interpretation of (A1.4).

A direct proof that $\tilde{\varphi}_n \approx \tilde{\varphi}_{n+1} \approx \tilde{\varphi}_{n+2}$ necessitates rather complicated formulae. In the first step with large R and ρ these formulae become simplified as follows.

For the sake of definiteness, let us assume that $\rho = O(\sqrt{n})$, $R = O(\sqrt{n})$ and neglect the error factors $1 + O(x^2)$. Then, a change $m \to m' = m+1$ in the first term of (A1.5) gives

$$R' = R/(1 + xR). \tag{A1.6}$$

Hence, the value of R becomes positive and small after a sufficient number of iterations. Similarly, the second item in (A1.5) transforms ρ in accord with (A1.4):

$$\rho' = \frac{\rho(1 + xR)}{1 + \rho x R}. \tag{A1.7}$$

The rate of convergence is quicker here and $\rho \to 1$ up to the higher-order corrections. The third term of (A1.5) preserves its form on the same $1 + O(x^2)$ level of precision, $1/(\rho R^2) \to 1/(\rho' R'^2)$. We may conclude that $\tilde{\varphi}_n \to O(1)$ (cf figure 1).

Beyond the above leading-order approximation, an accumulation (convergence) of the values h_n or $\tilde{\varphi}_n$ (near their fixed point) does not follow geometrically any more. Algebraic procedure necessitates a reintroduction of indices; the general ansatz (A1.5) ceases to be m independent in the higher precision. Indeed, the compatibility of (A1.5) with (A1.4) forces us to replace $1/(\rho R^2)$ by $1/(\rho R^2 + x)$ and we must modify even the simple formula (A1.6)

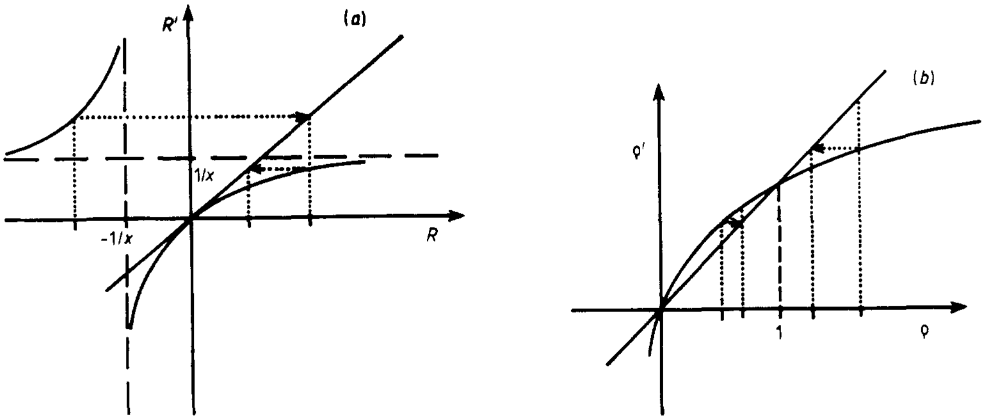

Figure 1. Geometric proof of the leading-order convergence (a) of the mapping $R \to R'$ in (A1.6), and (b) of the mapping $\rho \to \rho'$ in (A1.7).

$$R' = \frac{R}{1+xR} + \frac{x/R\rho}{1+xR} + \mathrm{O}(x^2). \tag{A1.8}$$

This is a good example of what becomes modified—a detailed analysis of (A1.8) shows that there still exists just one fixed point and the convergence takes place even up to the $R = \mathrm{O}(1)$ level of magnitude, $R \to \rho^{-1/3} > 0$. We see that R does not drop down to zero—the convergence assumption $\tilde{\varphi}_n \to 1$ is not contradictory.

In a small vicinity of the exact fixed points $\tilde{\varphi}_\infty = 1 - x/3 + \ldots$, we may employ the identities

$$\begin{aligned} \frac{\partial \tilde{\varphi}_{n+2}}{\partial \tilde{\varphi}_{n+1}} &= -\tilde{\varphi}_n \tilde{\varphi}^2_{n+2} = -\alpha^2 > -1 \\ \frac{\partial \tilde{\varphi}_{n+2}}{\partial \tilde{\varphi}_n} &= -\tilde{\varphi}_{n+1} \tilde{\varphi}^2_{n+2} = -\beta^2 > -1 \end{aligned} \tag{A1.9}$$

and, for the sufficiently small deviations $\mathrm{d}\tilde{\varphi}_n = \tilde{\varphi}_n - \tilde{\varphi}_\infty$, we write

$$\mathrm{d}\tilde{\varphi}_{n+2} = -\alpha^2\, \mathrm{d}\tilde{\varphi}_{n+1} - \beta^2\, \mathrm{d}\tilde{\varphi}_n. \tag{A1.10}$$

We may interpret these relations as recurrences:

$$\begin{pmatrix} 1 & & & \\ \alpha^2 & 1 & & \\ \beta^2 & \alpha^2 & 1 & \\ & \beta^2 & \alpha^2 & 1 \\ & & & \ldots \end{pmatrix} \begin{pmatrix} \mathrm{d}\tilde{\varphi}_{n+2} \\ \mathrm{d}\tilde{\varphi}_{n+3} \\ \vdots \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ \vdots \end{pmatrix}$$

$$c_1 = -\beta^2\, \mathrm{d}\tilde{\varphi}_n - \alpha^2\, \mathrm{d}\tilde{\varphi}_{n+1} \qquad c_2 = -\beta^2\, \mathrm{d}\tilde{\varphi}_{n+1}. \tag{A1.11}$$

When we invert the corresponding matrix

$$\begin{pmatrix} \mathrm{d}\tilde{\varphi}_{n+2} \\ \mathrm{d}\tilde{\varphi}_{n+3} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & \\ u & 1 & \\ & u & 1 \\ & & \ldots \end{pmatrix}^{-1} \begin{pmatrix} 1 & & \\ v & 1 & \\ & v & 1 \\ & & \ldots \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ \vdots \end{pmatrix}$$

$$u + v = \alpha^2 \approx 1 - x \qquad uv = \beta^2 \approx 1 - x \tag{A1.12}$$

$$u, v = \frac{1-x}{2} \pm \frac{i\sqrt{3}}{2}\left(1 - \frac{x}{3}\right) + O(x^2)$$

$$= \exp(\pm i\pi/3) - \frac{x}{\sqrt{3}} \exp(\pm i\pi/6) + O(x^2)$$

$$= \exp(\pm i\psi)\, e^{-\lambda} \qquad \psi, \lambda > 0$$

we get

$$d\tilde{\varphi}_{n+m+1} = (-1)^{m+1} \frac{u^m - v^m}{u - v} c_1 + (-1)^m \frac{u^{m-1} - v^{m-1}}{u - v} c_2$$

$$= \exp(-m\lambda) M(m) \qquad |M(m)| < M_0 < \infty \tag{A1.13}$$

i.e. $d\tilde{\varphi}_n \to 0$ for $n \to \infty$. Thus, in the vicinity of its real fixed point, the mapping (A1.4) generates the convergent sequences $\tilde{\varphi}_n$ from an arbitrary initial pair.

References

Barik N and Barik B K 1981 *Pramana* **17** 489
Biswas S N, Datta K, Saxena R P, Srivastava P K and Varma V S 1971 *Phys. Rev.* D **4** 3617-20
Chaudhuri R N and Mukherjee B 1984 *J. Phys. A: Math. Gen.* **17** 3327-34
Flessas G P 1982 *J. Phys. A: Math. Gen.* **15** L1-5
Ginsburg C A 1982 *Phys. Rev. Lett.* **48** 839-41
Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
Quigg C and Rosner J L 1979 *Phys. Rep.* **56** 167-235
Znojil M 1981 *Lett. Math. Phys.* **5** 405-9
—— 1982 *Phys. Rev.* D **26** 3750-1
—— 1983a *J. Phys. A: Math. Gen.* **16** 213-20
—— 1983b *J. Math. Phys.* **24** 1136-41